Crossed products of representable localization algebras

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Outline

G: second countable, locally compact group
X: proper G-space

$RL(\mathcal{H}_X)$: representable localization algebra (what is it?)

$RL(\mathcal{H}_X) \rtimes_r G$: (reduced) crossed product

!! $K_*(RL(\mathcal{H}_X) \rtimes_r G) \cong RK_*^G(X)$
(RHS: representable G-equivariant K-homology of X)

Applications to the Baum–Connes conjecture

Reference

- S. Nishikawa, Crossed product approach to equivariant localization algebras, Arxiv preprint, 2021
- R. Willett and G. Yu, Higher Index Theory, Cambridge Studies in Advanced Mathematics, Cambridge University Press, 2020.
- M. Dadarlat, R. Willett, and J. Wu. Localization C*-algebras and K-theoretic duality, Ann. K-Theory, 2018.

X-module

An X-module is a (separable) Hilbert space $\mathcal{H}_X$ equipped with a non-degenerate representation of the C^*-algebra $C_0(X)$.

If X is discrete set, $\mathcal{H}_X \cong \bigoplus_{x \in X} \mathcal{H}_x$.

For $T \in \mathcal{L}(\mathcal{H}_X)$, T has compact support if for a compact subset $K \subset X$

$$T = \chi_K T \chi_K$$

(χ_K: characteristic function of K).

representable localization algebra

$\mathcal{H}_X$: X-module

$RL(\mathcal{H}_X) \subset C_b([1,\infty), \mathcal{K}(\mathcal{H}_X))$ is the closure of

the algebra that consists of $[t \mapsto T_t]$ such that

- T_t $(t \geq 1)$ have uniform compact support,
- $\lim_{t\to\infty} |||[\phi, T_t]||| = 0$ for any $\phi \in C_0(X)$.

$RL(\mathcal{H}_X)$ contains the ideal $RL_0(\mathcal{H}_X) = C_0([1,\infty), \mathcal{K}(\mathcal{H}_X))$.

$$0 \to RL_0(\mathcal{H}_X) \to RL(\mathcal{H}_X) \to RL_Q(\mathcal{H}_X) \to 0$$

$RL_Q(\mathcal{H}_X)$ is a $C_0(X)$-algebra.

Examples

$X = point$
$\mathcal{H}_X = l^2(\mathbb{N})$

$$RL(\mathcal{H}_X) = C_b([1,\infty), \mathcal{K}(l^2(\mathbb{N}))).$$

The evaluation ev_1 at $t = 1$ induces

$$K_*(RL(\mathcal{H}_X)) \cong K_*(\mathcal{K}(l^2(\mathbb{N}))).$$

Examples

X: discrete set
$\mathcal{H}_X = l^2(X) \otimes l^2(\mathbb{N})$

$$RL(\mathcal{H}_X) = RL_0(\mathcal{H}_X) + \bigoplus_X C_b([1,\infty), \mathcal{K}(l^2(\mathbb{N})))$$

$$K_*(RL(\mathcal{H}_X)) \cong \bigoplus_X \mathbb{Z}$$

Representable K-homology

G: trivial
For each X, choose an ample X-module $\mathcal{H}_X$.
For each $f\colon X \to Y$, choose covering isometries $V_t^f\colon \mathcal{H}_X \to \mathcal{H}_Y$ $(t \geq 1)$.

Theorem (Willett and Yu)

The assignment

$$X \mapsto \mathbb{D}_*(X) = K_*(RL(\mathcal{H}_X)), \ \ [f\colon X \to Y] \mapsto \mathbb{D}_*(f) = \mathrm{Ad}_{V_t^f *}$$

is a functor from the category of locally compact spaces and continuous maps to the category of graded abelian groups. The functor is naturally equivalent to the representable K-homology

$$RK_*(X) \cong \varinjlim_{Y \subset X, \mathrm{cpt}} KK_*(C(Y), \mathbb{C}).$$

G-equivariant case

X: (proper) G-space

An X-G-module is a (separable) Hilbert space $\mathcal{H}_X$ equipped with a non-degenerate, covariant representation of the G-C^*-algebra $C_0(X)$.

Recall: $RL(\mathcal{H}_X) \subset C_b([1,\infty), \mathcal{K}(\mathcal{H}_X))$ is the closure of

the algebra that consists of $[t \mapsto T_t]$ such that

- T_t $(t \geq 1)$ have uniform compact support,
- $\lim_{t\to\infty} \|[\phi, T_t]\| = 0$ for any $\phi \in C_0(X)$,
- for non-discrete G, we require G-continuity for the function T.

We are interested in the (reduced) crossed product

$$RL(\mathcal{H}_X) \rtimes_r G.$$

Example

G: discrete
$H \subset G$: finite subgroup
$X = G/H$: discrete set
$\mathcal{H}_X = l^2(G/H) \otimes l^2(\mathbb{N})$

$$RL(\mathcal{H}_X) = RL_0(\mathcal{H}_X) + \bigoplus_{G/H} C_b([1,\infty), \mathcal{K}(l^2(\mathbb{N})))$$

$$K_*(RL(\mathcal{H}_X) \rtimes_r G) \cong K_*(C_0(G/H) \rtimes_r G) \cong K_*(C_r^*(H))$$

Representable G-equivariant K-homology

For each proper G-space X, choose a universal X-G-module $\mathcal{H}_X$.
For each G-map $f\colon X \to Y$, choose G-equivariant covering isometries $V_t^f\colon \mathcal{H}_X \to \mathcal{H}_Y$ $(t \geq 1)$.

Theorem

The assignment

$$X \mapsto \mathbb{D}_*^G(X) = K_*(RL(\mathcal{H}_X) \rtimes_r G), \;\; [f\colon X \to Y] \mapsto \mathbb{D}_*^G(f) = \mathrm{Ad}_{V_t^f *}$$

is a functor from the category of proper G-spaces and continuous G-maps to the category of graded abelian groups.
The functor is naturally equivalent to the representable G-equivariant K-homology

$$RK_*^G(X) \cong \varinjlim_{Y \subset X, \mathrm{Gcpt}} KK_*^G(C_0(Y), \mathbb{C}).$$

Forget-control map

$\mathcal{H}_X$: X-G-module
The evaluation map at $t = 1$

$$\mathrm{ev}_1 \colon RL(\mathcal{H}_X) \to \mathcal{K}(\mathcal{H}_X)$$

descends to

$$\mathrm{ev}_1 \colon RL(\mathcal{H}_X) \rtimes_r G \to \mathcal{K}(\mathcal{H}_X) \rtimes_r G.$$

It gives a homomorphism

$$\mathcal{F} = \mathrm{ev}_{1*} \colon \mathbb{D}_*^G(X) \to K_*(C_r^*(G)).$$

The Baum–Connes assembly map

Theorem

The homomorphism

$$\mathcal{F}\colon \mathbb{D}_*^G(X) \to K_*(C_r^*(G))$$

is naturally equivalent to the Baum–Connes assembly map

$$\mu_X^G\colon RK_*^G(X) \to K_*(C_r^*(G)).$$

(The statement generalizes to the case with coefficient G-C^*-algebra).

The Dirac dual–Dirac method in this context

For some X-G-module $\mathcal{H}_X$, we want

$$\nu\colon K_*(C_r^*(G)) \to K_*(RL(\mathcal{H}_X) \rtimes_r G)$$

which inverts the BC assembly map $\mu^G = \mathrm{ev}_{1*}$.
Such ν should be obtained from a cycle for $KK^G(\mathbb{C}, RL(\mathcal{H}_X))$.

More concretely, we want a pair $(\mathcal{H}_X, T)$ of a graded X-G-module $\mathcal{H}_X$ and an odd, self-adjoint (G-continuous) $T \in M(RL(\mathcal{H}_X))$ such that

$$1 - T^2,\ g(T) - T \in RL(\mathcal{H}_X)\ \ (g \in G).$$

The Dirac dual–Dirac method in this context

Theorem

If there is $(\mathcal{H}_X, T)$ (as before), such that $[\mathcal{H}_X, T_1] = 1_G$ in $KK^G(\mathbb{C}, \mathbb{C})$, the Baum–Connes conjecture (with coefficients) holds, i.e. the assembly map μ^G is an isomorphism (for all coefficients).

Remark: We can apply the theorem to prove BCC for groups G that act on a finite-dimensional CAT(0)-cubical space with bounded geometry, completely independently of the Higson–Kasparov Theorem (G is a-T-menable).

Thank you for your time!